12/10/23

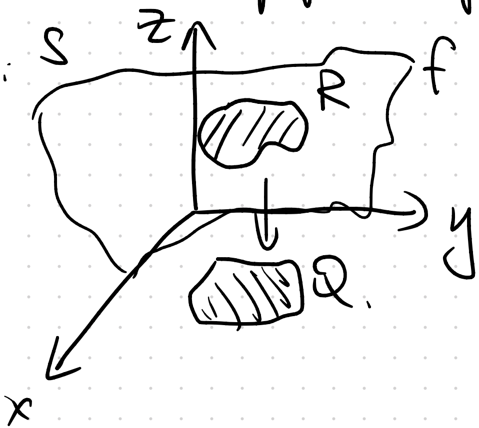
MATH4030 Tutorial

Announcements:

- HW1 Graded.
- HW2 due 16/10.
- Midterm is next Tuesday

Q1: Show that the area A of a bounded region R of the surface $z=f(x,y)$ is given by $A=\iint_Q \sqrt{1+f_x^2+f_y^2}\,dx\,dy$ where Q is the normal projection of R onto the xy plane.

Pf:



S is given by the param

$$X(x,y) = (x, y, f(x,y)).$$

$$A=\iint_R \sqrt{EG-F^2}\,dx\,dy.$$

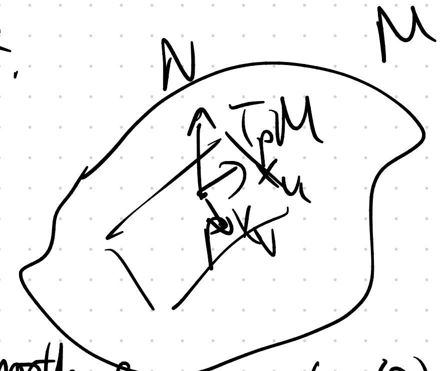
$$X_x = (1,0,f_x), \quad X_y=(0,1,f_y)$$

$E = \langle X_x, X_x \rangle = 1 + f_x^2$, $F = \langle X_x, X_y \rangle = f_x f_y$

$G = \langle X_y, X_y \rangle = 1 + f_y^2$.

$$EG - F^2 = (1+f_x^2)(1+f_y^2) - f_x^2 f_y^2 = 1 + f_x^2 + f_y^2. \quad //$$

<u>Recap</u>: An orientation N of a regular surface M is a smooth unit normal vector field.



- N smooth.
- N has image in S^2.
- $\forall p$, $N \perp T_pM$.
- $N = \dfrac{X_u \times X_v}{|X_u \times X_v|}$

The Shape Operator wrt. N <u>at p</u> is given by the following: let $v \in T_pM$, $\alpha(t)$ smooth curve w/ $\alpha(0) = p$, $\alpha'(0) = v$.

$$S_p(v) = -\frac{d}{dt} N(\alpha(t)) \Big|_{t=0}.$$ "differential of Gauss map".

$S_p(X_u) = -N_u$, $S_p(X_v) = -N_v$.

second fundamental form: $\mathrm{II}_p : T_pM \times T_pM \to \mathbb{R}$

$$\mathrm{II}_p(v,w) = g(S_p(v), w) = \langle S_p(v), w \rangle.$$

$\rightsquigarrow$ coefficients of 2nd ff. $\rightsquigarrow$ define Gauss curvature

Mean curvature.

Q2: Given param. of catenoid

$$X(u,v) = \left(c\cosh\left(\tfrac{v}{c}\right)\cos u,\ c\cosh\left(\tfrac{v}{c}\right)\sin u,\ v\right),\ c>0.$$

Compute N, $S_p(X_u)$, $S_p(X_v)$.

Sol'n: $X_u = \left(-c\cosh\left(\tfrac{v}{c}\right)\sin u,\ c\cosh\left(\tfrac{v}{c}\right)\cos u,\ 0\right)$

$X_v = \left(\sinh\left(\tfrac{v}{c}\right)\cos u,\ \sinh\left(\tfrac{v}{c}\right)\sin u,\ 1\right).$

$X_u \times X_v = \left(c\cosh\left(\tfrac{v}{c}\right)\cos u,\ c\cosh\left(\tfrac{v}{c}\right)\sin u,\ -c\cosh\left(\tfrac{v}{c}\right)\sinh\left(\tfrac{v}{c}\right)\right)$

$$|X_u \times X_v| = \left(c^2\cosh^2\left(\tfrac{v}{c}\right) + c^2\cosh^2\left(\tfrac{v}{c}\right)\sinh^2\left(\tfrac{v}{c}\right)\right)^{\frac{1}{2}} = c\cosh^2\left(\tfrac{v}{c}\right).$$

$$\left(1+\sinh^2 = \cosh^2\right).$$

$$N = \left(\frac{\cos u}{\cosh^2\left(\frac{v}{c}\right)}, \frac{\sin u}{\cosh\left(\frac{v}{c}\right)}, -\tanh\left(\tfrac{v}{c}\right)\right).$$

$$S_p(X_u) = -N_u = \left(\frac{\sin u}{\cosh\left(\frac{v}{c}\right)}, \frac{-\cos u}{\cosh\left(\frac{v}{c}\right)}, 0\right).$$

$$S_p(X_v) = -N_v = \frac{1}{c}\left(\cos u \tanh\left(\tfrac{v}{c}\right)\operatorname{sech}\left(\tfrac{v}{c}\right), \sin(u)\tanh\left(\tfrac{v}{c}\right)\operatorname{sech}\left(\tfrac{v}{c}\right), \operatorname{sech}^2\left(\tfrac{v}{c}\right)\right).$$

$\rightsquigarrow$ find coeff's of 2nd ff. $\rightsquigarrow$ compute mean curvature $H \equiv 0$.

Catenoid is an example of a "minimal surface".

minimal surfaces are critical points of area functional $\int_S dA$